S. C. Kleene and R. E. Vesley The Foundations of Intuitionistic Mathematics North-Holland, 1965

(b) $Q_1(y, \alpha(y)) \vee Q_2(y, \alpha(y)), \neg(Q_1(y, \alpha(y)) \,\&\, Q_2(y, \alpha(y))) \vdash^{y\alpha}$

$$\exists\alpha[\alpha(0)=q \,\&\, \forall y\alpha(y')=\begin{cases} r_1(y, \alpha(y)) \textit{ if } Q_1(y, \alpha(y)), \\ r_2(y, \alpha(y)) \textit{ if } Q_2(y, \alpha(y))]. \end{cases}$$

(c) $Q_1(\bar\alpha(y)) \vee Q_2(\bar\alpha(y)), \neg(Q_1(\bar\alpha(y)) \,\&\, Q_2(\bar\alpha(y))) \vdash^{y\alpha}$

$$\exists\alpha\forall y\alpha(y)=\begin{cases} r_1(\bar\alpha(y)) \textit{ if } Q_1(\bar\alpha(y)), \\ r_2(\bar\alpha(y)) \textit{ if } Q_2(\bar\alpha(y)) \end{cases}$$

and

$Q_1(y, \tilde\alpha(y)) \vee Q_2(y, \tilde\alpha(y)), \neg(Q_1(y, \tilde\alpha(y)) \,\&\, Q_2(y, \tilde\alpha(y))) \vdash^{y\alpha}$

$$\exists\alpha\forall y\alpha(y)=\begin{cases} r_1(y, \tilde\alpha(y)) \textit{ if } Q_1(y, \tilde\alpha(y)), \\ r_2(y, \tilde\alpha(y)) \textit{ if } Q_2(y, \tilde\alpha(y)). \end{cases}$$

PROOFS. (a) In the SPECIAL CASE that $Q_1(y)$, $Q_2(y)$ are prime formulas, or equivalent to prime formulas by applications of #D and #E, we need only apply Lemma 5.3 (a) for p(y) the term p obtained by using #F with $\vdash Q_i(y) \sim q_i=0$ and with $p_i(y)$ as the p_i ($i = 1, 2$).

However, the GENERAL CASE can be treated directly, thus. The first assumption formula gives two cases. CASE 1: $Q_1(y)$. Then $\neg Q_2(y)$. So

$$(Q_1(y) \vee Q_2(y)) \,\&\, (Q_1(y) \supset p_1(y)=p_1(y)) \,\&\, (Q_2(y) \supset p_1(y)=p_2(y)).$$

By ∃-introd.,

$$\exists a[(Q_1(y) \vee Q_2(y)) \,\&\, (Q_1(y) \supset a=p_1(y)) \,\&\, (Q_2(y) \supset a=p_2(y))].$$

CASE 2: $Q_2(y)$. Similarly. — By ∀-introd. and *2.2,

$$\exists\alpha\forall y[(Q_1(y) \vee Q_2(y)) \,\&\, (Q_1(y) \supset \alpha(y)=p_1(y)) \,\&\, (Q_2(y) \supset \alpha(y)=p_2(y))].$$

(b) Substituting $(y)_0$, $\lambda t(y)_1$ for y, α, and using *0.1: $Q_1((y)_0, (y)_1) \vee Q_2((y)_0, (y)_1)$ and $\neg(Q_1((y)_0, (y)_1) \,\&\, Q_2((y)_0, (y)_1))$. So, using (a), we can assume for ∃-elim.

(i) $$\forall y\rho(y)=\begin{cases} r_1((y)_0, (y)_1) \text{ if } Q_1((y)_0, (y)_1), \\ r_2((y)_0, (y)_1) \text{ if } Q_2((y)_0, (y)_1). \end{cases}$$

Applying Lemma 5.3 (b) with $\rho(\langle y, z\rangle)$ as the r(y, z), assume for ∃-elim.

(ii) $$\alpha(0)=q \,\&\, \forall y\alpha(y')=\rho(\langle y, \alpha(y)\rangle).$$

Taking $\langle y, \alpha(y)\rangle$ for y in (i) (by ∀-elim.) and using *25.1, the result with (ii) gives $Q_i(y, \alpha(y)) \supset \alpha(y')=r_i(y, \alpha(y))$. By &-, ∀- and ∃-introds., we obtain the required formula. It does not contain ρ or α free, so the ∃-elims. can be completed.

(c) WITH $\bar\alpha(y)$. By *158 since Seq(z) is prime, Seq(z) ∨ ¬Seq(z). Using cases thence, and in the first case *23.6 with $Q_1(\bar\alpha(y)) \vee Q_2(\bar\alpha(y))$: (i) $(\text{Seq}(z) \,\&\, Q_1(z)) \vee (\text{Seq}(z) \,\&\, Q_2(z)) \vee \neg\text{Seq}(z)$. Using *23.6 with $\neg(Q_1(\bar\alpha(y)) \,\&\, Q_2(\bar\alpha(y)))$: $\text{Seq}(z) \supset \neg(Q_1(z) \,\&\, Q_2(z))$. Using this and *50, the three cases in (i) are mutually exclusive. Assume for ∃-elim. from the result of an application of (a) with $m = 3$,

$$\forall z\rho(z)=\begin{cases} r_1(z) & \text{if } \text{Seq}(z) \,\&\, Q_1(z), \\ r_2(z) & \text{if } \text{Seq}(z) \,\&\, Q_2(z), \\ 0 & \text{if } \neg\text{Seq}(z). \end{cases}$$

Now use Lemma 5.3 (c) with ρ(z) as the r(z) (and later *23.5).

(c) WITH $\tilde\alpha(y)$. Apply (c) with $\bar\alpha(y)$, for $Q_i(\text{lh}(z), \Pi_{i<\text{lh}(z)} p_i^{(z)_i \dot- 1})$, $r_i(\text{lh}(z), \Pi_{i<\text{lh}(z)} p_i^{(z)_i \dot- 1})$ as the $Q_i(z)$, $r_i(z)$.

LEMMA 5.6. *Let* x *be a variable, and* A(x) *a formula. Then*

$$\exists! x A(x) \vdash A(x) \vee \neg A(x).$$

PROOF. Assume preparatory to ∃-elim. from ∃!wA(w), A(w) & ∀x(A(x) ⊃ w=x). By *158, w=x ∨ w≠x. CASE 1: w=x. Then A(x), whence A(x) ∨ ¬A(x). CASE 2: w≠x. Then ¬A(x), whence again A(x) ∨ ¬A(x).

§ 6. Postulate on spreads (the bar theorem). 6.1. In the intuitionistic set theory or analysis of Brouwer, a fundamental role is played by what he called a "set (Menge)" in his early papers on the subject (**1918–9** I p. 3, **1919** pp. 204–205 or 950–951, **1924–7** I pp. 244–245) and more recently a "spread" (**1954** p. 8). There are several versions of the notion of 'spread', differing in details. We begin with a version differing from that of Brouwer's early papers, reproduced in Kleene 1950a § 1 (p. 680 end line 8, add "> 0"), by the omission of what Brouwer called "sterilized (gehemmt)" sequences. In 6.9, we shall consider other versions.

A given *spread* is generated by (i) *choosing* natural numbers in sequence, (either freely or) under an effective restriction which says, given the (numbers chosen in the respective) previous choices if any and any number, whether that number may be chosen next, and (ii) after each choice *correlating* effectively an object (depending on the previous choices if any and that choice) from a fixed countable set. Furthermore, under (i) it is effectively determined after each choice whether (depending on the previous choices if any and that choice) the sequence of choices is to *terminate* therewith or shall continue; in the latter case, the restriction governing the choices must allow at least one natural number to be chosen next.

When a sequence of choices terminates, the *element* of the set or spread correlated to the sequence is the finite sequence of the objects correlated to the choices up to its termination. When a sequence of choices continues unterminated ad infinitum, the *element* correlated to the sequence is the infinite sequence of the objects correlated to the choices; intuitionistically this element is not considered as completed, but only as in process of growth as the choices proceed.

The word "effectively" in the foregoing is intended to convey what Brouwer expressed (in his early papers) by speaking of a "law (Gesetz)"; and indeed in **1924** § 1, **1927** § 2 he used "algorithm (Algorithmus)" in a related connection. What choices are permitted, and whether termination takes place, is determined by a law, which we call the *choice law*. What object is correlated is determined by another law, which we call the *correlation law*. (Cf. Kleene 1950a p. 680, and Heyting **1956** p. 34, where the terminology is a little different.) These two laws each operate upon the finite sequence of the choices (natural numbers) up to and including the one which is under consideration (i.e. the natural number about to be chosen, when the question is whether the choice of it after the choices already made if any is permissible; the one just chosen, when the question is whether the sequence of choices thereupon terminates, or what object is thereupon correlated).

A set or spread is not thought of intuitionistically as the "totality" of its elements, not even in the case all (permitted) choice sequences terminate so that the elements themselves become intuitionistically completed objects. To do so would (in general) involve the completed infinite (IM p. 48); e.g. the spread in which all choice sequences terminate after one choice which is completely free, with the number chosen correlated, is simply the set of all (unit sequences of) natural numbers. A spread from the intuitionistic standpoint is the pair of laws governing the generation process under which its elements grow. Through his notion of 'spread', Brouwer found a way, while maintaining the standpoint of the potential infinite, to deal with collections

some of which are even uncountably infinite (of classical cardinal number $2^{\aleph_0}$).

The objects correlated to the choices in Brouwer's applications may be, e.g., natural numbers, rational numbers, intervals with rational endpoints. Since for a given spread they must be chosen from a given countable class of objects, abstractly we can always take them to be natural numbers. When we do so, the notations available in the formal system suffice for the theory of spreads.

Indeed, these notions include the fundamental constituents for dealing with spreads. These constituents can be combined under the formation rules of the system in a flexible manner, so that the particular way of combining them that gives a spread loses some of its preeminence in this formalism. Cf. however 7.8 below.

6.2. In this section, we shall concentrate on the choice sequences, which may underlie a spread, and which can be regarded as themselves constituting a spread by taking for the correlation law the trivial one which correlates the last natural number chosen. If then there is no restriction on the choices, the spread consists simply of all the infinite sequences of natural numbers in process of growth. This Brouwer called the *universal spread*. We study it now.

When exactly t (≥ 0) natural numbers $a_0, a_1, \ldots, a_{t-1}$ have been chosen successively, we have in other words chosen the first t values $\alpha(0), \alpha(1), \ldots, \alpha(t-1)$ of a number-theoretic function $\alpha(x)$, the remaining values of which are still undetermined. Now we may associate with any finite sequence $a_0, \ldots, a_{t-1}$ of natural numbers the natural number $a = p_0^{a_0+1} \cdot \ldots \cdot p_{t-1}^{a_{t-1}+1} = \langle a_0+1, \ldots, a_{t-1}+1 \rangle = [a_0, \ldots, a_{t-1}] = \bar{\alpha}(t)$, whereupon $t = \mathrm{lh}(\bar{\alpha}(t))$, and $a_i = \alpha(i) = (\bar{\alpha}(t))_i \dot{-} 1$ for $i < t$ (cf. #18–#23 in 5.5, and 5.7). This maps the finite sequences of choices 1–1 onto the natural numbers a such that $\mathrm{Seq}(a)$, which we call *sequence numbers*. The theory of choice sequences can now be dealt with in terms of the sequence numbers. The fundamental relation between sequence numbers is that of a sequence number a to the sequence numbers $a*2^{s+1}$ ($s = 0, 1, 2, \ldots$) which represent the sequences $a_0, \ldots, a_{t-1}, s$ coming from $a_0, \ldots, a_{t-1}$ by choosing one more number s; the numbers $a*2^{s+1}$ are thus exactly the numbers $\bar{\alpha}(t+1)$ for the various functions α such that $a = \bar{\alpha}(t)$.

6.3. Because the sequences of choices ("Wahlfolgen" in Brouwer **1918–9** and **1924–7**, "infinitely proceeding sequences" in Brouwer **1952** and Heyting **1956**) are considered intuitionistically as in process of growing by new choices, especial prominence is given in intuitionism to those properties of choice sequences which if possessed can be recognized effectively as possessed at some (finite) stage in the growth of the choice sequence. Such a property of a choice sequence α is of the form $(Ex)R(\bar{\alpha}(x))$ where $R(a)$ is a number-theoretic predicate, effective at least when applied to sequence numbers a.

With respect to such a predicate $R(a)$, we say that, as a choice sequence $\alpha(0), \alpha(1), \alpha(2), \ldots$ is generated, the finite sequence of the choices $\alpha(0), \ldots, \alpha(t-1)$, or the sequence number $\bar{\alpha}(t)$ representing these first t choices, is *secured*, if it is known already from these t choices by the test of the predicate R that α possesses the property $(Ex)R(\bar{\alpha}(x))$, i.e. if $(Ex)_{x\leq t}R(\bar{\alpha}(x))$; *past secured*, if this was known already without the last choice, i.e. if $(Ex)_{x<t}R(\bar{\alpha}(x))$; *immediately secured*, if this is known only after the last choice, i.e. if $(x)_{x<t}\bar{R}(\bar{\alpha}(x))$ & $R(\bar{\alpha}(t))$ (the first conjunctive member can be omitted if R is taken so that, for any α, $R(\bar{\alpha}(x))$ is true of at most one x). We say $\alpha(0), \ldots, \alpha(t-1)$, or $\bar{\alpha}(t)$, is *securable*, if, no matter how the future choices (the $t+1$-st, $t+2$-nd, $t+3$-rd, ...) are made, α will possess the property $(Ex)R(\bar{\alpha}(x))$, i.e. if $(\beta)[\bar{\beta}(t)=\bar{\alpha}(t) \rightarrow (Ex)R(\bar{\beta}(x))]$ or equivalently $(Ex)_{x<t}R(\bar{\alpha}(x)) \vee (\beta)(Ex)R(\bar{\alpha}(t)*\bar{\beta}(x))$. In particular (changing the bound variable β to α) 1 is securable exactly if $(\alpha)(Ex)R(\bar{\alpha}(x))$. We have stated these notions with respect to a fixed predicate $R(a)$. A sequence number w not past secured is securable with respect to $R(a)$ exactly if 1 is securable with respect to $\lambda a\, R(w*a)$. A sequence number w is *barred*, if 1 is securable with respect to $\lambda a\, R(w*a)$, i.e. if $(\alpha)(Ex)R(w*\bar{\alpha}(x))$. (Numbers other than sequence numbers are to be *unsecured, unsecurable, unbarred.*)

6.4. We have used function variables in expressing these notions. But there is a basic difference between the classical and the intuitionistic concepts; for the intuitionists, the functions are not completed. The universal function quantifier (β) or (α), with its scope, in the expression for securability cannot be considered intuitionistically as a conjunction extended over all completed one-place number-theoretic functions, as it is classically. The intuitionistic meaning of $(\alpha)(Ex)R(\bar{\alpha}(x))$ is that, whenever one chooses successively natural numbers $\alpha(0), \alpha(1), \alpha(2), \ldots$ in any way, one must eventually encounter an x such that $R(\bar{\alpha}(x))$.

How then can the intuitionists utilize the notion of securability? To begin with, they can particularize, compatibly with their interpretation of (α), from $(\alpha)(Ex)R(\bar{\alpha}(x))$ to $(Ex)R(\bar{\alpha}_1(x))$ for such particular choice sequences α_1 as they can specify; these, in connection with which Brouwer (**1952** p. 143, **1954** p. 7) uses the term "sharp arrows", are ones whose growth can be completely governed in advance by a law (after any $t \geq 0$ choices, the law allows exactly one next choice). We have the formal counterpart of this in Axiom Schema 10F, where the functors u express primitive recursive functions in the case they contain no function variables (by Lemma 3.3).

But it would seem that this makes rather weak use of $(\alpha)(Ex)R(\bar{\alpha}(x))$. In fact, under the interpretation that an α_1 giving a sharp arrow is a general recursive function, $(\alpha_1)(Ex)R(\bar{\alpha}_1(x))$ is in general weaker than $(\alpha)(Ex)R(\bar{\alpha}(x))$; and the important "fan theorem" (in 6.10 below) fails when its hypothesis is weakened in the corresponding manner (Kleene 1950a § 3, or Lemma 9.8 below). The intuitionists may refrain from adopting this interpretation, but they are in no position to refute it, since their actual constructions or laws conform to it (Chapter II below). Pursuing the matter further from the classical standpoint, while the fan theorem becomes true upon enlarging the class of α's to the arithmetical functions (those such that $\alpha(x)=w$ is an arithmetical predicate IM p. 239; cf. Lemma 9.12 below), in order to exhaust the full force of $(\alpha)(Ex)R(\bar{\alpha}(x))$ not even all the hyperarithmetical functions suffice (Kleene **1955b** pp. 210, 208 with **1959** p. 48; or **1959b**).

REMARK 6.1. In the intuitionistic system, using *158, we can prove $\forall\alpha\forall x(\alpha(x)=0 \vee \alpha(x)\neq 0)$, which seems to imply that any function α taking only 0 and 1 as values is recursive. (More generally, we can prove $\forall\alpha\forall x\forall w(\alpha(x)=w \vee \alpha(x)\neq w)$, which seems to say that, for each α, the predicate $\alpha(x)=w$ is decidable, so presumable recursive, so by IM Theorem III p. 279 the function α, $= \lambda x\, \mu w\alpha(x)=w$, is recursive.) On the other hand, as noted, we cannot interpret the universal quantifier (α) to mean "for all recursive functions α" without making the fan theorem of intuitionism false. This apparent contradiction is explained thus. As we choose the numbers $\alpha(0), \alpha(1), \alpha(2), \ldots$ making up any choice sequence α, it will be known after each choice what number has been chosen; it is in this sense that $\forall\alpha\forall x(\alpha(x)=0 \vee$

$\alpha(x) \neq 0)$ is true. But as a choice sequence $\alpha(0), \alpha(1), \alpha(2), \ldots$ grows, in advance of each choice any number in the case of the universal spread (any number ≤ 1 in the case of the spread of choice sequences governed by $(x)\alpha(x) \leq 1$) is eligible to be chosen; so the α is not restricted to be a recursive function.

REMARK 6.2. In the present classical system with the same formation rules as the intuitionistic, the functors u available for Axiom Schema 10F are the same. (This is not so in classical systems like the ones in Hilbert-Bernays 1939 Supplement IV having a choice operator ε or descriptive operator ι.) Fuller use of assumptions $\forall\alpha A(\alpha)$ is obtained in the classical system via indirect proofs.

6.5. Brouwer found a solution to the problem of how to utilize an hypothesis of securability more fully than by Axiom Schema 10F. This consists in looking at the situation from the opposite direction, proceeding backwards from those sequence numbers $\bar{\alpha}(x)$ for which $R(\bar{\alpha}(x))$ to the other sequence numbers having such numbers in all their (sufficiently continued) extensions.

To fix our ideas, let us confine our attention for the moment to sequence numbers not past secured (so that, in any sequence α of choices, we don't overrun the first x at which we find $R(\bar{\alpha}(x))$ true). Then, slightly paraphrasing Brouwer **1927** FOOTNOTE 7 to make it read in our notation and terminology: Thought through intuitionistically, this securability is nothing else than the property which is defined thus. It holds for every sequence number a such that $R(a)$. It holds for any sequence number a, if for every s $(s = 0, 1, 2, \ldots)$ it holds for $a*2^{s+1}$. This remark draws after it immediately the well-orderedness property ....

In other words, Brouwer's Footnote 7 says that securability is that property (of sequence numbers not past secured) which originates at the immediately secured sequence numbers, and propagates back to the unsecured but securable numbers across the junctions between a sequence number a and its immediate extensions $a*2^{s+1}$ $(s = 0, 1, 2, \ldots)$.

Let us review the situation using a geometrical picture (Figure 1). We can represent the universal spread 6.2 by a "tree", with the sequence numbers $a = p_0^{a_0+1}\cdot\ldots\cdot p_{t-1}^{a_{t-1}+1} = [a_0, \ldots, a_{t-1}]$ at the vertices. The initial (leftmost) vertex is occupied by the sequence number $1 = [\,] = \bar{\alpha}(0)$. From any vertex, occupied by the sequence number a,

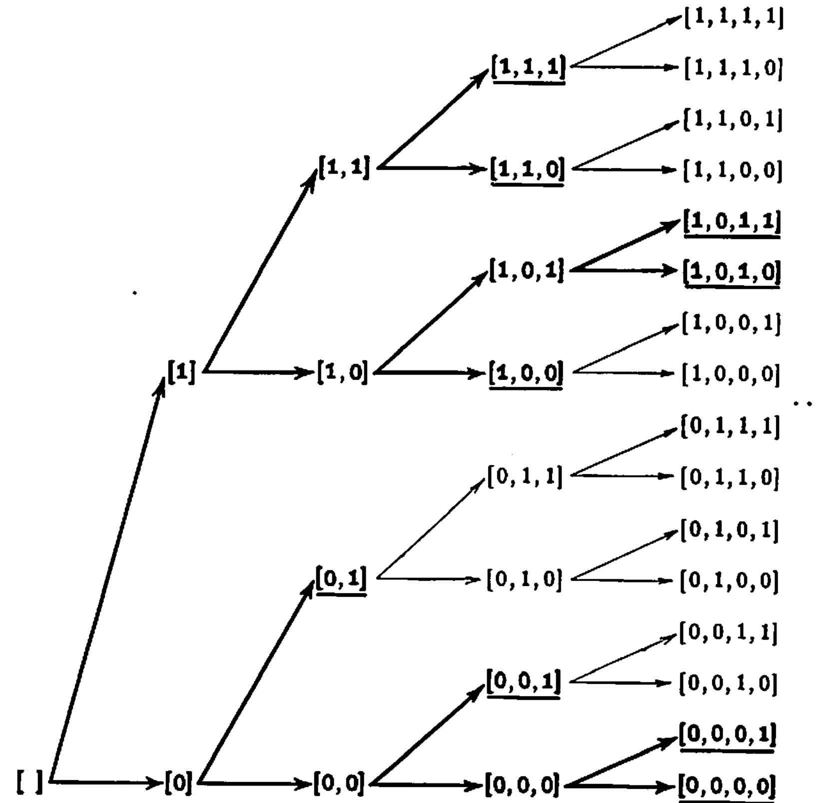

Figure 1.

infinitely many arrows lead to the next vertices, occupied by the sequence numbers $a*2^{s+1}$ $(s = 0, 1, 2, \ldots)$. A part of this tree is shown in Figure 1; but the arrows for $s > 1$ are left to our imagination, as well as the vertices for $t = \text{lh}(a) > 4$ suggested by the dots. (The figure actually shows the "binary spread" or "binary fan" 6.10 as far as its vertices with $\text{lh}(a) \leq 4$.)

An infinite choice sequence α or $\alpha(0), \alpha(1), \alpha(2), \ldots$ is represented by an infinite path in the tree, starting at the leftmost vertex (occupied by) [ ] and following arrows; a finite sequence of choices by an initial segment of such a path, or by the vertex $\bar{\alpha}(t)$ at the (right) end of that segment. Thus, before $\alpha(0)$ is chosen, we are at the vertex [ ]; then if we choose $\alpha(0) = 1$, we move to the vertex [1]; choosing next $\alpha(1) = 0$, we continue to [1, 0]; choosing $\alpha(2) = 1$, to [1, 0, 1]; choosing $\alpha(3) = 1$, to [1, 0, 1, 1]; etc.

Consider a predicate $R(a)$, effective at least when applied to sequence numbers a. For each α, let us follow the corresponding path in the tree (starting from [ ]) until we first encounter a vertex $\bar{\alpha}(x)$ for which $R(\bar{\alpha}(x))$, if we ever do, whereupon we underline that vertex. In the language of 6.3, we underline (the vertices occupied by) the immediately secured sequence numbers.

Now $(\alpha)(Ex)R(\bar{\alpha}(x))$, as we considered it in 6.3 and (intuitionistically) in 6.4, means geometrically that, along each infinite path starting from the leftmost vertex [ ] and following arrows, we will encounter an underlined vertex. This is illustrated in Figure 1, so far as it can be shown with only the arrows for $s = 0, 1$. More generally, $(\beta)(Ex)R(a*\bar{\beta}(x))$ or in words *a is securable* (*but not past secured*) means geometrically that, along each infinite path starting from the vertex occupied by a and following the arrows, we will encounter an underlined vertex.

Brouwer's reversal of the direction consists in replacing this meaning of *a is securable* (*but not past secured*) by that of belonging to the class of sequence numbers which is defined to include the ones underlined, and to include a whenever it includes all $a*2^{s+1}$ for $s = 0, 1, 2, \ldots$, but to include no other sequence numbers. (This definition is an example of an inductive definition, in the terminology of IM § 53.)

In Figure 1, the securable but not past secured sequence numbers are those which are in bold face (heavy type), if we suppose appropriate behavior along paths containing arrows with $s \geq 2$. But under the first meaning of *securable* (*but not past secured*), which we now call the *explicit sense*, a vertex's being in bold face means that proceeding rightward from it in the direction of arrows along all possible divergent paths an underlined vertex will be encountered. Under the second meaning (the *inductive sense*), a vertex's being in bold face signifies its membership in the class of vertices generated by putting into the class the underlined vertices, and proceeding in the leftward or convergent direction (reverse to arrows) to include a in the class whenever all $a*2^{s+1}$ ($s = 0, 1, 2, \ldots$) are included in the class.

Since the explicit sense, which our symbols in 6.3 directly express, had already been used before Brouwer's **1927** Footnote 7 was introduced, that Footnote 7 must come to this: The two meanings of *securable* (*but not past secured*) are equivalent; and this equivalence is given by intuition (by thinking the matter through intuitionistically). We agree with him.

In the figure, whether one puts vertices in bold face by the criterion of finding an underlined vertex at or to the right of them along all paths, or moves leftward across the figure putting vertices in bold face by the two principles generating a class of vertices, the result is the same.

One of the implications in this equivalence is actually unproblematical, i.e. easily proved (cf. end 6.7). The other implication, that by *securable* (*but not past secured*) in the explicit sense of *securable* (*but not past secured*) in the inductive sense (or of the "well-orderedness property", which the latter entails immediately) is essentially what Brouwer subsequently called the "bar theorem" (**1954** p. 14, cf. Remark 6.3 below).

In **1924** § 1 (cf. **1924a** §§ 1, 2), in the text of **1927** § 2, and in **1954**, he used a more complicated analysis to prove the bar theorem. Footnote 7 of **1927** concluded, "The proof carried through in the text for the latter property [well-orderedness] seems to me nevertheless of interest on account of the propositions included in its line of thought."

We shall simply introduce what is needed here by an axiom schema *26.3 which gives the effect of the bar theorem for the case of the universal spread. This schema takes the form of a principle of induction which attributes to the securable (but not past secured) sequence numbers any property expressible in the symbolism of the system which originates and propagates in the same way as the securability property itself (under the inductive sense). Our procedure amounts to adopting Brouwer's **1927** Footnote 7 in place of the more elaborate treatment in the text of **1927**.

We thus quickly get over a moot point in Brouwer's deduction of his analysis by postulating an axiom schema. This may strike some as an evasion. But this axiom schema is independent of the other intuitionistic postulates, as we shall see in Corollary 9.9 (and 9.2, by which its negation is unprovable). So there can be a question of deriving the axiom schema (the bar theorem), only if we first substitute another postulate to derive it from. We are unconvinced that any known substitute is more fundamental and intuitive. However, in view of the attention which the proof in Brouwer's text of **1927** has continued to receive, we shall also examine that, in 6.12.

6.6. We consider now just how to state the bar theorem in the formal symbolism.

The definition of a property in Brouwer's Footnote 7 reads, under the restriction there to sequence numbers not past secured, as an inductive definition of the securable (sequence) numbers. If we substitute "a which is secured" for "a such that $R(a)$", then without the restriction it reads as an inductive definition of all the securable numbers (Kleene **1955a** p. 416). If we omit the restriction, but require R to be a predicate such that, for any α, $R(\bar{\alpha}(x))$ for at most one x, it reads as an inductive definition of the numbers securable but not past secured. If we simply omit the restriction, it reads as an inductive definition of the barred numbers. It makes little difference to us here which reading we use, and the last is the simplest.

We also obtain some simplification by stating the induction principle corresponding to the inductive definition only for inferring properties of 1 (for which 'securable', 'securable but not past secured' and 'barred' are equivalent). We do not lose thereby, as we shall verify in 6.11.

For securability in the explicit sense of 6.3 we now write "securable$_E$", in the inductive sense of Footnote 7 "securable$_I$". The bar theorem is then the implication

(*) $$\text{securable}_E \rightarrow \text{securable}_I,$$

when the right side is rendered by the principle of induction corresponding to the inductive definition (cf. IM § 53). We want to formalize this, applied to 1, with respect to R.

The left side of (*) is then simply $(\alpha)(Ex)R(\bar{\alpha}(x))$.

Let $\mathfrak{J}(R, A)$ be $(a)[\text{Seq}(a) \,\&\, R(a) \rightarrow A(a)] \,\&\, (a)[\text{Seq}(a) \,\&\, (s)A(a*2^{s+1}) \rightarrow A(a)] \rightarrow A(1)$; and for any formulas A(a) and R(a), let $\mathfrak{J}$(R, A) be the correspondingly constructed formula. The principle of induction rendering the right side of (*) is $(A)\mathfrak{J}(R, A)$.

Thus we render (*) in informal symbolism as $(\alpha)(Ex)R(\bar{\alpha}(x)) \rightarrow (A)\mathfrak{J}(R, A)$. Expressing this in the formal symbolism as nearly as we can in the absence of predicate variables (cf. IM p. 432), we are led to $\forall\alpha\exists x R(\bar{\alpha}(x)) \supset \mathfrak{J}(R, A)$, which (trivially rearranged) is *26.1.

6.7. Before postulating a slight restriction of this for the basic system or the intuitionistic system, we verify that it is provable in the classical system. The proof is a formalization of the classical proof of (*) in Kleene **1955a** (E) p. 417.

If a, s, x are any number variables (a and s distinct), α is any function variable, A(a) is any formula not containing s free in which s is free for a, and R(a) is any formula not containing α or x free in which α and x are free for a: Bar Theorem (classical)

*26.1°. $\vdash \forall\alpha\exists x R(\bar{\alpha}(x)) \,\&\, \forall a[\text{Seq}(a) \,\&\, R(a) \supset A(a)] \,\&\, \forall a[\text{Seq}(a) \,\&\, \forall s A(a*2^{s+1}) \supset A(a)] \supset A(1)$.

PROOF. By the classical propositional calculus, it will suffice to assume

(a) $\forall a[\text{Seq}(a) \,\&\, R(a) \supset A(a)]$,
(b) $\forall a[\text{Seq}(a) \,\&\, \forall s A(a*2^{s+1}) \supset A(a)]$,
(c) $\neg A(1)$,

and deduce $\neg\forall\alpha\exists x R(\bar{\alpha}(x))$, which by the classical predicate calculus (*85, *86) is equivalent to $\exists\alpha\forall x\neg R(\bar{\alpha}(x))$. Likewise (b) is equivalent to $\forall a[\text{Seq}(a) \,\&\, \neg A(a) \supset \exists s\neg A(a*2^{s+1})]$, whence by *97 $\forall a\exists s[\text{Seq}(a) \,\&\, \neg A(a) \supset \neg A(a*2^{s+1})]$, whence by *2.2 $\exists\sigma\forall a[\text{Seq}(a) \,\&\, \neg A(a) \supset \neg A(a*2^{\sigma(a)+1})]$. Assume for $\exists$-elim. from this

(d) $\forall a[\text{Seq}(a) \,\&\, \neg A(a) \supset \neg A(a*2^{\sigma(a)+1})]$.

By Lemma 5.3 (c), $\exists\alpha\forall x\alpha(x)=\sigma(\bar{\alpha}(x))$; so assume

(e) $\forall x\alpha(x)=\sigma(\bar{\alpha}(x))$.

Now we deduce by induction

(f) $\neg A(\bar{\alpha}(x))$.

BASIS. By *23.1 and *B3, $\bar{\alpha}(0)=1$. So by (c), $\neg A(\bar{\alpha}(0))$. IND. STEP. $\bar{\alpha}(x') = \bar{\alpha}(x)*2^{\alpha(x)+1}$ [*23.8] $= \bar{\alpha}(x)*2^{\sigma(\bar{\alpha}(x))+1}$ [(e)]. So by (d) with the hyp. ind. and *23.5, $\neg A(\bar{\alpha}(x'))$. – By (f) and (a) with *23.5, $\neg R(\bar{\alpha}(x))$. By $\forall$- and $\exists$-introd., $\exists\alpha\forall x\neg R(\bar{\alpha}(x))$.

The converse implication

(**) $$\text{securable}_I \rightarrow \text{securable}_E$$

(Kleene **1955a** (D) p. 416) is $(A)\mathfrak{J}(R, A) \rightarrow (\alpha)(Ex)R(\bar{\alpha}(x))$. This holds intuitionistically, a fortiori from $\mathfrak{J}(R, A_1) \rightarrow (\alpha)(Ex)R(\bar{\alpha}(x))$ where $A_1 = \lambda a\ (\alpha)(Ex)R(a*\bar{\alpha}(x))$. So *26.2 can be considered as giving (**) in the basic system.

If a, s, x are any distinct number variables, α is any function variable, and R(a) is any formula not containing x, s, α free in which x, s, α are free for a:

*26.2. $\vdash \{\forall a[\mathrm{Seq}(a) \,\&\, R(a) \supset \forall\alpha\exists x R(a*\bar{\alpha}(x))] \,\&$
$\forall a[\mathrm{Seq}(a) \,\&\, \forall s\forall\alpha\exists x R((a*2^{s+1})*\bar{\alpha}(x)) \supset \forall\alpha\exists x R(a*\bar{\alpha}(x))]$
$\supset \forall\alpha\exists x R(1*\bar{\alpha}(x))\} \supset \forall\alpha\exists x R(\bar{\alpha}(x))$.

PROOF. Using $a=a*\bar{\alpha}(0)$ (by *22.6 with $^{\times}$23.1, *B3),

(a) $\forall a[\mathrm{Seq}(a) \,\&\, R(a) \supset \forall\alpha\exists x R(a*\bar{\alpha}(x))]$.

Toward (b) below, assume **(i)** Seq(a) and **(ii)** $\forall s\forall\alpha\exists x R((a*2^{s+1})*\bar{\alpha}(x))$. Using (ii), $\exists x R((a*2^{\alpha(0)+1})*\overline{\{\lambda x\alpha(1+x)\}}(x))$. Assume preparatory to $\exists$-elim. **(iii)** $R((a*2^{\alpha(0)+1})*\overline{\{\lambda x\alpha(1+x)\}}(x))$. But $(a*2^{\alpha(0)+1})*\overline{\{\lambda x\alpha(1+x)\}}(x)$ $= a*(2^{\alpha(0)+1}*\overline{\{\lambda x\alpha(1+x)\}}(x))$ [*22.9 with (i), *22.5, *23.5] $=$ $a*\bar{\alpha}(1+x)$ [23.7 with $^{\times}$23.1, *B4, *B3, *127]. So by $\exists$-introd., (completing) the $\exists$-elim., and $\forall$-introd., $\forall\alpha\exists x R(a*\bar{\alpha}(x))$. By &-elim. and $\supset$- and $\forall$-introd.,

(b) $\forall a[\mathrm{Seq}(a) \,\&\, \forall s\forall\alpha\exists x R((a*2^{s+1})*\bar{\alpha}(x)) \supset \forall\alpha\exists x R(a*\bar{\alpha}(x))]$.

Assuming the antecedent of the main implication of *26.2, and using (a) and (b), we obtain $\forall\alpha\exists x R(1*\bar{\alpha}(x))$, whence the consequent $\forall\alpha\exists x R(\bar{\alpha}(x))$ follows by *22.7 with *23.5.

6.8. The restriction that *R* be an effective predicate, introduced beginning 6.3 (but immaterial from the classical standpoint), must be made explicit in postulating the bar theorem (*) for the basic system or the intuitionistic system. As expressed by *26.1 simply, (*) is inconsistent with the further intuitionistic postulate $^{\times}$27.1 to be introduced in § 7, by *27.23. We give four forms $^{\times}$26.3a–$^{\times}$26.3d of the new axiom schema. Whichever one is introduced now as the postulate, all axioms by each of the others become provable. When it is immaterial which one we cite, we call it simply $^{\times}$26.3. The stipulations for $^{\times}$26.3a and $^{\times}$26.3c are the same as for *26.1. For $^{\times}$26.3b, α and ρ are any distinct function variables, etc.

$^{\times}$26.3a. $\forall a[\mathrm{Seq}(a) \supset R(a) \vee \neg R(a)] \,\&\, \forall\alpha\exists x R(\bar{\alpha}(x)) \,\&$
$\forall a[\mathrm{Seq}(a) \,\&\, R(a) \supset A(a)] \,\&\, \forall a[\mathrm{Seq}(a) \,\&\, \forall s A(a*2^{s+1}) \supset A(a)]$
$\supset A(1)$.

$^{\times}$26.3b. $\forall\alpha\exists x \rho(\bar{\alpha}(x))=0 \,\&$
$\forall a[\mathrm{Seq}(a) \,\&\, \rho(a)=0 \supset A(a)] \,\&\, \forall a[\mathrm{Seq}(a) \,\&\, \forall s A(a*2^{s+1}) \supset A(a)]$
$\supset A(1)$.

$^{\times}$26.3c. $\forall\alpha\exists! x R(\bar{\alpha}(x)) \,\&$
$\forall a[\mathrm{Seq}(a) \,\&\, R(a) \supset A(a)] \,\&\, \forall a[\mathrm{Seq}(a) \,\&\, \forall s A(a*2^{s+1}) \supset A(a)]$
$\supset A(1)$.

$^{\times}$26.3d. $\forall\alpha\exists x[R(\bar{\alpha}(x)) \,\&\, \forall y_{y<x}\neg R(\bar{\alpha}(y))] \,\&$
$\forall\alpha\forall x[R(\bar{\alpha}(x)) \,\&\, \forall y_{y<x}\neg R(\bar{\alpha}(y)) \supset A(\bar{\alpha}(x))] \,\&$
$\forall a[\mathrm{Seq}(a) \,\&\, \forall s A(a*2^{s+1}) \supset A(a)] \supset A(1)$.

DERIVATION OF $^{\times}$26.3b FROM $^{\times}$26.3a. Taking R(a) in $^{\times}$26.3a as $\rho(a)=0$, we have $R(a) \vee \neg R(a)$ by *158, a fortiori $\forall a[\mathrm{Seq}(a) \supset R(a) \vee \neg R(a)]$.

$^{\times}$26.3a FROM $^{\times}$26.3b. Assume the four hypotheses **(a)–(d)** of $^{\times}$26.3a. By *158, because Seq(a) is prime, $\mathrm{Seq}(a) \vee \neg\mathrm{Seq}(a)$. Using cases thence, and in the first case subcases from (a), $(\mathrm{Seq}(a) \,\&\, R(a)) \vee \neg(\mathrm{Seq}(a) \,\&\, R(a))$. Using this with *50 to apply Lemma 5.5 (a), assume preparatory to $\exists$-elim. from the result

$$\forall a \rho(a) = \begin{cases} 0 & \text{if } \mathrm{Seq}(a) \,\&\, R(a), \\ 1 & \text{if } \neg(\mathrm{Seq}(a) \,\&\, R(a)). \end{cases}$$

Now $\mathrm{Seq}(a) \supset (R(a) \sim \rho(a)=0)$, using which and *23.5 the three hypotheses of $^{\times}$26.3b follow from (b)–(d).

$^{\times}$26.3c FROM $^{\times}$26.3a. Assume $\forall\alpha\exists! x R(\bar{\alpha}(x))$. Assume Seq(a), so via *23.6 we can put $a=\bar{\alpha}(x)$ (i.e. we assume this preparatory to $\exists$-elims.). Using Lemma 5.6, $R(\bar{\alpha}(x)) \vee \neg R(\bar{\alpha}(x))$, whence $R(a) \vee \neg R(a)$. By (completing) the $\exists$-elims., $\supset$- and $\forall$-introd., $\forall a[\mathrm{Seq}(a) \supset R(a) \vee \neg R(a)]$. Also, $\forall\alpha\exists x R(\bar{\alpha}(x))$.

$^{\times}$26.3a FROM $^{\times}$26.3c. Assume the four hyps. **(a)–(d)** of $^{\times}$26.3a. Let R′(a) be $R(a) \,\&\, \forall y_{y<\mathrm{lh}(a)}\neg R(\Pi_{i<y} p_i^{(a)_i})$, so using *23.5 and *23.4 $R'(\bar{\alpha}(x)) \sim R(\bar{\alpha}(x)) \,\&\, \forall y_{y<x}\neg R(\bar{\alpha}(y))$. By *23.5 $\mathrm{Seq}(\bar{\alpha}(x))$, so (a) gives $R(\bar{\alpha}(x)) \vee \neg R(\bar{\alpha}(x))$. Thence by *149a and *174b $\forall\alpha[\exists x R(\bar{\alpha}(x)) \supset \exists! x R'(\bar{\alpha}(x))]$, and by *69 $\forall\alpha\exists x R(\bar{\alpha}(x)) \supset \forall\alpha\exists! x R'(\bar{\alpha}(x))$. So we have $\forall\alpha\exists! x R'(\bar{\alpha}(x))$. Since $R'(a) \supset R(a)$, we also have $\forall a[\mathrm{Seq}(a) \,\&\, R'(a) \supset A(a)]$. Now we can apply $^{\times}$26.3c with R′ as the R.

$^{\times}$26.3d FROM $^{\times}$26.3c. Use *174b.

6.9. The foregoing induction principle $^{\times}$26.3 takes care of the bar theorem for the universal spread. We should like it also for other spreads of choice sequences.

So instead of dealing with the class of all the sequence numbers *a*, characterized by Seq(*a*), we shall now deal with any suitable subclass of them, which we shall characterize by $\sigma(a)=0$ for some function σ.

For simplicity, we may omit from consideration terminated sequences of choices (cf. 6.1), so this σ will serve as the choice law (the other function of the choice law in 6.1, to say when a sequence of choices terminates, is suppressed). We may do this here without loss, since we are interested only in what happens up to an x such that $R(\bar{\alpha}(x))$. Indeed in general, with a simplified choice law σ that doesn't provide for termination, we can still obtain the effect of termination, either (a) by using a predicate R and considering $\alpha(0), \alpha(1), \alpha(2), \ldots$ to terminate at $\alpha(x-1)$ for the least x if any such that $R(\bar{\alpha}(x))$, or (b) for spreads with a non-trivial correlation law, by using positive integers as (or to represent) the objects which we are interested in correlating, and correlating 0 otherwise (essentially Brouwer **1924–7** I Footnote 1).

In our theory of choice sequences we have been using to advantage the empty sequence, represented by the sequence number $\bar{\alpha}(0) = 1$. (Brouwer employed neither the empty sequence, nor sequence numbers.) For spreads all of whose elements are to be sequences with the same first member, we find it convenient to correlate that first member to the empty choice sequence. Then the correlation law ρ operates simply on all sequence numbers a with $\sigma(a) = 0$. When we don't want the elements all to begin with the same first member (correlated to 1), we may simply ignore what $\rho(1)$ is. But whether we do or do not wish to consider $\rho(1)$ as first member of the elements, it seems to us natural to take advantage of our empty sequence by letting the spread be non-empty exactly when the empty sequence is permitted, i.e. when $\sigma(1) = 0$. Thus the choice law suffices itself for deciding whether a spread is empty or not.

When we thus both omit terminated sequences and use the empty sequence to test for a spread's not being empty, we are led to the following formula $\mathrm{Spr}(\sigma)$ expressing in the formal symbolism the restrictions on σ that it characterize the choice sequences for a spread.

$$\begin{aligned}\mathrm{Spr}(\sigma):\quad & \forall a[\sigma(a)=0 \supset \mathrm{Seq}(a)] \mathbin{\&} \forall a[\sigma(a)=0 \supset \exists s\sigma(a*2^{s+1})=0] \\ & \mathbin{\&} \forall a[\mathrm{Seq}(a) \mathbin{\&} \sigma(a)>0 \supset \forall s\sigma(a*2^{s+1})>0].\end{aligned}$$

In *26.4 we state the bar theorem for spreads generally, using this version of the notion of 'spread'. The second hypothesis $\sigma(1)=0$ expresses that the spread is not empty.

If we were simply to omit terminated sequences (which would give the version of Heyting **1956** pp. 34–35, = essentially Brouwer **1924–7** I Footnote 2), we would use instead of $\mathrm{Spr}(\sigma)$ the formula $\mathrm{Spd}(\sigma)$ obtained from it by prefixing $\sigma(1)=0\ \&$ and replacing the second $\forall a$ by $\forall a_{a>1}$. *26.4 would become *26.4′ with $\exists s\sigma(2^{s+1})=0$ replacing $\sigma(1)=0$ to express the non-emptiness of the spread.

Under the version of 'spread' in Brouwer **1954**, all spreads are non-empty.

That a choice sequence α is permitted by the choice law σ of a spread is expressed formally by $\forall x\sigma(\bar{\alpha}(x))=0$, which we abbreviate as "$\alpha\in\sigma$".

The form *26.4a of *26.4 corresponds to ˣ26.3a and is proved from it; using instead ˣ26.3b–ˣ26.3d, corresponding forms *26.4b–*26.4d are obtained (not written out when clear). Also, from any one of *26.4a–*26.4d the others can be derived (using only Postulate Groups A–D), as with *26.3. Similarly with *26.6, *26.7 and *26.8 below.

*26.4a. $\vdash \mathrm{Spr}(\sigma) \mathbin{\&} \sigma(1)=0 \mathbin{\&} \forall a[\sigma(a)=0 \supset R(a) \vee \neg R(a)] \mathbin{\&}$
$\forall \alpha_{\alpha\in\sigma} \exists x R(\bar{\alpha}(x)) \mathbin{\&} \forall a[\sigma(a)=0 \mathbin{\&} R(a) \supset A(a)] \mathbin{\&}$
$\forall a[\sigma(a)=0 \mathbin{\&} \forall s\{\sigma(a*2^{s+1})=0 \supset A(a*2^{s+1})\} \supset A(a)] \supset A(1).$

*26.4d. $\vdash \mathrm{Spr}(\sigma) \mathbin{\&} \sigma(1)=0 \mathbin{\&} \forall \alpha_{\alpha\in\sigma} \exists x[R(\bar{\alpha}(x)) \mathbin{\&} \forall y_{y<x} \neg R(\bar{\alpha}(y))] \mathbin{\&}$
$\forall \alpha \forall x[\sigma(\bar{\alpha}(x))=0 \mathbin{\&} R(\bar{\alpha}(x)) \mathbin{\&} \forall y_{y<x} \neg R(\bar{\alpha}(y)) \supset A(\bar{\alpha}(x))] \mathbin{\&}$
$\forall a[\sigma(a)=0 \mathbin{\&} \forall s\{\sigma(a*2^{s+1})=0 \supset A(a*2^{s+1})\} \supset A(a)] \supset A(1).$

PROOF OF *26.4a. In I, we shall set up a mapping of the universal spread onto the spread characterized by σ. Thus, to each element α of the universal spread, the function α_γ $(= \lambda t\, (\gamma(\bar{\alpha}(t')))_t \dot{-} 1)$ will belong to the spread σ, as shown by (ε). If α already belongs to σ, $\alpha_\gamma = \alpha$, as shown by (η). (We give (ζ) and (η) for use in proving *26.7a, *27.4 etc.) In II, this mapping carries the bar theorem for the universal spread into the bar theorem for σ.

I. Assume the first two hypotheses of *26.4a, call them **(1)** and **(2)**. By cases from $\sigma(a)=0 \vee \sigma(a)\neq 0$ (by *158), using (1), $\exists s[\sigma(a)=0 \supset \sigma(a*2^{s+1})=0]$, whence by $\forall$-introd. and *2.2 $\exists\pi\forall a[\sigma(a)=0 \supset \sigma(a*2^{\pi(a)+1})=0]$. Assume

(α) $\quad \forall a[\sigma(a)=0 \supset \sigma(a*2^{\pi(a)+1})=0].$

In the following formula (β), the case hypotheses are exhaustive (by cases from applications of *158, since the components are prime) and mutually exclusive (using *50). So Lemma 5.5 (c) applies (indeed, the special case), and we assume (preparatory to $\exists$-elim. from the result)

$$(\beta)\quad \forall a\gamma(a)=\begin{cases}0 \text{ if } \neg\mathrm{Seq}(a),\\ 1 \text{ if } \mathrm{Seq}(a)\ \&\ \mathrm{lh}(a)=0,\\ (\tilde{\gamma}(a))_B*2^{S+1} \text{ if } \mathrm{Seq}(a)\ \&\ \mathrm{lh}(a)\neq 0\ \&\ \sigma((\tilde{\gamma}(a))_B*2^{S+1})=0,\\ (\tilde{\gamma}(a))_B*2^{\pi((\tilde{\gamma}(a))_B)+1}\\ \qquad\qquad \text{ if } \mathrm{Seq}(a)\ \&\ \mathrm{lh}(a)\neq 0\ \&\ \sigma((\tilde{\gamma}(a))_B*2^{S+1})\neq 0\end{cases}$$

where B is $\Pi_{i<\mathrm{lh}(a)\dot{-}1}p_i^{(a)_i}$ and S is $(a)_{\mathrm{lh}(a)\dot{-}1}\dot{-}1$. If in (β) we use $\bar{\alpha}(0)$ for a (via ∀-elim.), the second case applies and gives $\gamma(\bar{\alpha}(0)) = \gamma(1) = 1$ (using *23.5, ˣ23.1, *B3). If in (β) we use $\bar{\alpha}(x')$ for a, then the third or fourth case applies; furthermore using *23.4, *23.2, *23.8 etc., $B=\bar{\alpha}(x)$, $S=\alpha(x)$, $\bar{\alpha}(x') = a = B\cdot p_x^{S+1} = B*2^{S+1}=\bar{\alpha}(x)*2^{\alpha(x)+1}$, so $B < a$ (using *143b, *3.10 etc.) and $(\tilde{\gamma}(a))_B = \gamma(B) = \gamma(\bar{\alpha}(x))$ (by *24.2). Now by ind., using (2) in the basis, and (α) to deal with the fourth case of (β) in the ind. step,

$$(\gamma)\quad \sigma(\gamma(\bar{\alpha}(x)))=0.$$

Let "α_γ" abbreviate $\lambda t(\gamma(\bar{\alpha}(t')))_t\dot{-}1$. Now we deduce by induction

$$(\delta)\quad \overline{\alpha_\gamma}(x)=\gamma(\bar{\alpha}(x)).$$

Basis: trivial. Ind. step. $\overline{\alpha_\gamma}(x') = \overline{\alpha_\gamma}(x)*2^{((\gamma(\bar{\alpha}(x')))_x\dot{-}1)+1}$ [*23.8, ˣ0.1] $= \gamma(\bar{\alpha}(x))*2^{((\gamma(\bar{\alpha}(x')))_x\dot{-}1)+1}$ [hyp. ind.], which (using $(\gamma(\bar{\alpha}(x))*2^{A+1})_x = (\overline{\alpha_\gamma}(x)*2^{A+1})_x$ [hyp. ind.] $= A+1$), if the third case of (β) applies to $a=\bar{\alpha}(x')$, $= \gamma(\bar{\alpha}(x))*2^{\alpha(x)+1} = \gamma(\bar{\alpha}(x'))$ {if the fourth case applies, $= \gamma(\bar{\alpha}(x))*2^{\pi(\gamma(\bar{\alpha}(x)))+1} = \gamma(\bar{\alpha}(x'))$}. — By (γ) and (δ),

$$(\varepsilon)\quad \alpha_\gamma\in\sigma.$$

We also deduce by induction

$$(\zeta)\quad \sigma(\bar{\alpha}(x))=0 \supset \gamma(\bar{\alpha}(x))=\bar{\alpha}(x).$$

Ind. step. Assuming $\sigma(\bar{\alpha}(x'))=0$, the third member of (1) gives $\sigma(\bar{\alpha}(x))=0$, so by hyp. ind. $\gamma(\bar{\alpha}(x))=\bar{\alpha}(x)$, and the third case of (β) applies. — By (δ), (ζ), *23.2 and *6.3, $\sigma(\bar{\alpha}(x'))=0 \supset \alpha_\gamma(x)=\alpha(x)$, whence

$$(\eta)\quad \alpha\in\sigma \supset \alpha_\gamma=\alpha.$$

II. Assume also the remaining hyps. **(3)–(6)** of *26.4a. We shall apply ˣ26.3a with $R(\gamma(a))$, $A(\gamma(a))$ as the $R(a)$, $A(a)$. If we can then verify the four hyps. of ˣ26.3a, the concl. of *26.4a will follow using $\gamma(1)=1$ (in I). We get the first hyp. by (γ) with (3) (using *23.6 to put $a=\bar{\alpha}(x)$ preparatory to ∃-elims.). For the second, by (ε) and (4)

$\exists xR(\overline{\alpha_\gamma}(x))$, whence by (δ) $\exists xR(\gamma(\bar{\alpha}(x)))$. We get the third (putting $a=\bar{\alpha}(x)$) by (γ) with (5). For the fourth, assume $\mathrm{Seq}(a)\ \&\ \forall sA(\gamma(a*2^{s+1}))$. By (γ) with *23.6, $\sigma(\gamma(a))=0$. Put $x=\mathrm{lh}(a)$. Assuming $\sigma(\gamma(a)*2^{s+1})=0$, and using *22.8, *22.5, *23.6 to put $a*2^{s+1}=\bar{\alpha}(y)$ (then $y=x'$ [*22.8, *20.3, *23.5], $a\cdot p_x^{s+1} = a*2^{s+1}$ [ˣ21.1 etc.] $= \bar{\alpha}(x') = \bar{\alpha}(x)\cdot p_x^{\alpha(x)+1}$ [*23.8], so $s=\alpha(x)$ [*19.11, *22.2, *19.9, *6.3] and $a=\bar{\alpha}(x)$ [*133]), the third case of (β) applies to $\bar{\alpha}(x')$ and gives $\gamma(\bar{\alpha}(x'))=\gamma(a)*2^{s+1}$, so $\forall sA(\gamma(a*2^{s+1}))$ gives $A(\gamma(a)*2^{s+1})$; thus $\forall s\{\sigma(\gamma(a)*2^{s+1})=0 \supset A(\gamma(a)*2^{s+1})\}$. By (6), $A(\gamma(a))$.

6.10. From his bar theorem Brouwer inferred his "fan theorem" (implicit in **1923a** p. 4 (II); **1924** Theorem 2; **1927** Theorem 2; **1954** § 5). A "finite set" or "finitary spread", most recently called a *fan*, is a spread in which each choice must be from a finite collection of numbers. Say e.g. that, for $t = 0, 1, 2, \ldots$, the number $\alpha(t)$ must be chosen from among $0, 1, \ldots, \beta(\bar{\alpha}(t))$; i.e. $(t)\alpha(t)\leq\beta(\bar{\alpha}(t))$. We shall here be considering only the choice sequences underlying a fan, which constitute a fan by taking for the correlation law ρ the trivial correlation $\rho(\bar{\alpha}(x')) = \alpha(x)$. According to one version of the fan theorem (classically true), if, for all choice sequences α restricted to this fan (determined by β), $(Ex)R(\bar{\alpha}(x))$, then there is a finite upper bound z to the least x's for which $R(\bar{\alpha}(x))$. In this "pure" version, symbolized by *26.6a (or *26.6b–*26.6d), we can prove the fan theorem from the bar theorem with no further postulate. Another version *27.7 (classically false), favored by Brouwer, will follow from this by the new intuitionistic postulate ˣ27.1 of § 7. A classical contrapositive of the present version is König's lemma **1926**, which we shall give in Remark 9.11.

First, we give a proof of the present version of the fan theorem informally. Consider any sequence number a belonging to the given fan, i.e. representing a finite choice sequence belonging to that fan; by the *subfan issuing from a* we mean the fan of those choice sequences α by which a can be extended in the given fan, i.e. such that, for each x, the sequence number $a*\bar{\alpha}(x)$ represents a finite choice sequence belonging to that fan. We apply Brouwer's **1927** Footnote 7 in 6.5 above, but considering only sequence numbers not past secured belonging to the given fan: "for every s ($s = 0, 1, 2, \ldots$)" becomes "for every $s \leq \beta(a)$". We use the corresponding form of induction to prove as follows that, under the hyp. of the fan theorem for the given fan and the given predicate R, the conclusion of the fan theorem

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9.1, 9.2, *Theorem* 9.3 (c) *and* (d), *and Corollaries* 9.4 *and* 9.6 *hold* (call them LEMMA C/8.2 etc.).

PROOF. We reëxamine the former proofs, omitting the case of ×26.3 in that of Theorem 9.3, to verify that the reasoning holds good when the universe of functions is C.

LEMMA 9.8, toward Corollary 9.9. (Kleene 1950a § 3.) *There is a primitive recursive predicate* $R(a)$ *such that, writing* $\alpha \in \mathbf{0} \equiv \{\alpha$ *is general recursive*$\}$ (Kleene-Post **1954**) *and* $B(\alpha) \equiv (t)\alpha(t) \leq 1$:

(a) $(\alpha)_{\alpha\in 0 \& B(\alpha)}(Ex)R(\bar{\alpha}(x))$;

(b) $(z)(E\alpha)_{\alpha\in 0 \& B(\alpha)}(x)_{x\leq z}\bar{R}(\bar{\alpha}(x))$,

whence

(c) $\overline{(Ez)}(\alpha)_{\alpha\in 0 \& B(\alpha)}(Ex)_{x\leq z}R(\bar{\alpha}(x))$,

whence by the fan theorem (*the informal analog of* *26.6a)

(d) $\overline{(\alpha)}_{B(\alpha)}(Ex)R(\bar{\alpha}(x))$.

PROOF. Using the W_0, W_1 of IM p. 308, let

$$W(i, t, y) \equiv \begin{cases} W_0(t, y) & \text{if } i = 0, \\ W_1(t, y) & \text{if } i \neq 0, \end{cases}$$

$$R(a) \equiv (Et)_{t<\mathrm{lh}(a)}(Ey)_{y<\mathrm{lh}(a)\dot{-}t}W((a)_t \dot{-} 1, t, y).$$

Then, for each α with $B(\alpha)$,

(1) $R(\bar{\alpha}(x)) \equiv (Et)_{t<x}(Ey)_{y<x\dot{-}t}W_{\alpha(t)}(t, y)$.

(a) Consider any general recursive α with $B(\alpha)$. Using IM Theorem IV p. 281,

(2) $\alpha(t)=1 \equiv (Ey)T_1(f_0, t, y) \equiv (Ey)T_1((f)_0, t, y)$,
(3) $\alpha(t)=0 \equiv (Ey)T_1(f_1, t, y) \equiv (Ey)T_1((f)_1, t, y)$,

for suitable numbers $f_0, f_1, f = \langle f_0, f_1\rangle$. CASE 1: $\alpha(f) = 1$. Then $(Ey)T_1((f)_0, f, y)$; and $\overline{(Ey)}T_1((f)_1, f, y)$, whence $(z)\overline{T}_1((f)_1, f, z)$. So $(Ey)W_1(f, y)$, i.e. $(Ey)W_{\alpha(f)}(f, y)$. CASE 2: $\alpha(f) = 0$. Similarly. — By (1) (with $x = f+y+1$, $t = f$), $(Ex)R(\bar{\alpha}(x))$.

(b) Consider any z. Let

$$\alpha(t) = \begin{cases} 1 & \text{if } t<z \ \& \ (Ey)_{y<z\dot{-}t}W_0(t, y) \ \text{(CASE A)}, \\ 0 & \text{if } t<z \ \& \ (Ey)_{y<z\dot{-}t}W_1(t, y) \ \text{(CASE B)}, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\alpha\in\mathbf{0}$ & $B(\alpha)$. Consider any $x \leq z$. Suppose $R(\bar{\alpha}(x))$. By (1), there is a $t < x \leq z$ and a $y < x \dot{-} t \leq z \dot{-} t$ such that $W_{\alpha(t)}(t, y)$. Thence we obtain a contradiction, by cases. CASE 1: $\alpha(t) = 1$. Then $W_1(t, y)$, and by Case B of the definition of α, $\alpha(t) = 0$. CASE 2: $\alpha(t) = 0$. Similarly.

COROLLARY 9.9. *The bar theorem* ×26.3 *and the fan theorem* *26.6 *or* *27.7 *do not hold in the intuitionistic system without the former as axiom schema, i.e. some formulas of the forms* ×26.3, *26.6, *27.7 (*and via deducibility relationships*, *26.4, *26.7–×26.9, *27.8–*27.14) *are unprovable in it.*

Also, by Corollary 9.5, the negation of no instance of ×26.3 etc. is unprovable. So ×26.3 etc. are "independent" of the other postulates of the intuitionistic system.

PROOF OF COROLLARY 9.9. Taking C = {the general recursive functions} = **0** in Theorem 9.7, all formulas provable in the system in question are **0**/realizable.

×26.3, *26.6. We shall show that the following substitution instance of the fan theorem *26.6a (deducible in this system from an instance of ×26.3a) is not **0**/realizable: $\forall a[\mathrm{Seq}(a) \supset R(a) \vee \neg R(a)]$ & $\forall\alpha_{B(\alpha)}\exists x R(\bar{\alpha}(x)) \supset \exists z \forall \alpha_{B(\alpha)}\exists x_{x\leq z}R(\bar{\alpha}(x))$, where $B(\alpha)$ is $\forall t\alpha(t)\leq 1$ ($\lambda t 1$ being substituted for β in *26.6), $R(a)$ is a formula numeralwise expressing the primitive recursive predicate $R(a)$ of Lemma 9.8 obtained by the method of proof of Lemma 8.5, and for simplicity $x\leq z$ and $\alpha(t)\leq 1$ are prime. Suppose a general recursive function ϵ **0**/realizes it. By Lemma **0**/8.4a (ii) in Theorem 9.7, $R(\bar{\alpha}(x)) \to \{\epsilon_{R(\bar{\alpha}(x))}$ **0**/realizes-α, x $R(\bar{\alpha}(x))\}$. By Lemma 8.8, $\vdash R(a) \vee \neg R(a)$, whence $\vdash \forall a[\mathrm{Seq}(a) \supset R(a) \vee \neg R(a)]$. By Theorem **0**/9.3 (a), the latter formula is **0**/realized by a general recursive function ζ_0. Consider any $\alpha \in \mathbf{0}$; by Lemma **0**/8.4a (i), a function $\rho \in \mathbf{0}$ **0**/realizes-α $B(\alpha)$ only when $B(\alpha)$, and in this case by Lemma 9.8 (a) $x_1 = \mu x R(\bar{\alpha}(x))$ is defined and $R(\bar{\alpha}(x_1))$. So $\zeta_1 = \Lambda\alpha\,\Lambda\rho\,\langle x_1, \epsilon_{R(\bar{\alpha}(x_1))}\rangle$ **0**/realizes $\forall\alpha_{B(\alpha)}\exists x R(\bar{\alpha}(x))$. Hence $\eta = (\{\epsilon\}[\zeta_0, \zeta_1])_1$ **0**/realizes-z $\forall\alpha_{B(\alpha)}\exists x_{x\leq z}R(\bar{\alpha}(x))$ for $z = (\{\epsilon\}[\zeta_0, \zeta_1](0))_0$. Now consider any general recursive α such that $B(\alpha)$. By Lemma **0**/8.4a (ii), $B(\alpha)$ is **0**/realized-α by $\epsilon_{B(\alpha)}$; so $x\leq z$ is **0**/realized-x, z by $(\{\{\eta\}[\alpha]\}[\epsilon_{B(\alpha)}])_{1,0}$, and $R(\bar{\alpha}(x))$ is **0**/realized-α, x by $(\{\{\eta\}[\alpha]\}[\epsilon_{B(\alpha)}])_{1,1}$, for $x = (\{\{\eta\}[\alpha]\}[\epsilon_{B(\alpha)}](0))_0$. Then $x \leq z$, and by Lemma **0**/8.4a (i), $R(\bar{\alpha}(x))$. Thus $(\alpha)_{\alpha\in 0 \& B(\alpha)}(Ex)_{x\leq z}R(\bar{\alpha}(x))$, contradicting Lemma 9.8 (c).

*27.7. Proceeding similarly, for $R(\bar{\alpha}(b))$ as the $A(\alpha, b)$ of *27.7,